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LETTER TO THE EDITOR

Discrete third-order spectral problems and a new Toda-type equation

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Abstract
In this paper we construct the class of equations associated with a discrete spectral problem of third order. Among the equations of this class we find an integrable differential equation on the lattice with exponential nonlinearity. By considering its associated Darboux transformation we construct an explicit solution.

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The Toda lattice equation was the first nonlinear dynamical equation on the lattice which was proven to be integrable. Toda [1] showed that the equation was associated with a discretization of the Schrödinger spectral problem, the spectral problem connected to the best known integrable nonlinear evolution equation, the Korteweg–de Vries equation [2]. A few years later other integrable equations on the lattice were found: the discrete Nonlinear Schrödinger equation [3], the discrete sine–Gordon equation [4], the discrete Chiral field equation [5], etc, whose associated Lax pairs are written mainly in terms of matrices of rank 2. These equations are discretizations of continuous nonlinear partial differential equations but are also very important in themselves as many physical problems exist which have an intrinsic discrete nature.

In the continuous case, few integrable equations, such as the nonlinear Schrödinger equation, turn out to be very important as they govern the asymptotic behaviour, obtained via multiscale reductive perturbation techniques, of a large number of physical models [6, 7]. On the contrary, perturbation theory, when some of the variables vary on a lattice, is not so well developed; some tentative results are known [8] but more work on this needs to be done [9].

The derivation of new nonlinear integrable equations is always very important. The Gelfand–Dickii hierarchies [10] are the most natural extension of the Schrödinger spectral problem and contain many very important integrable nonlinear partial differential equations, such as the Boussinesq, the Sawada–Kotera and Kaup–Kupershmidt equations. So it seems natural to investigate their differential–difference counterparts. A partial analysis, though not complete, in this direction has been carried out by Blaszak and collaborators [11, 12], who considered fundamentally the Hamiltonian structure of some equations of the Gelfand–Dickii hierarchies and, in the case of completely discrete systems, by Nijhoff *et al* [13].

In this letter we consider the simplest discrete spectral problem belonging to the Gelfand–Dickii class, the extension of the discrete Schrödinger spectral problem:

$$L\psi_n = \lambda\psi_n \tag{1}$$

with

$$L = E^2 + A_n(t)E + B_n(t) + C_n(t)E^{-1} \tag{2}$$

where E is the shift operator in the variable n, defined by $E^j f_n = f_{n+j}$. Starting from (1), using the by now standard Lax technique [2,14], we can obtain many of the algebraic properties of the L operator as the recursive operator which provides us with the class of associated nonlinear differential–difference equations and the infinity of symmetries [15] and the Bäcklund and Darboux transformations.

Imposing the following boundary conditions on the fields A_n, B_n and C_n,

$$\lim_{n\to\pm\infty} A_n(t) = 0 \qquad \lim_{n\to\pm\infty} B_n(t) = 0 \qquad \lim_{n\to\pm\infty} C_n(t) = 1 \tag{3}$$

and applying the Lax technique, we get the following class of nonlinear differential difference equations:

$$\begin{pmatrix} \frac{\mathrm{d}A_n(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}B_n(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}C_n(t)}{\mathrm{d}t} \end{pmatrix} = f_1(\mathcal{L}) \begin{pmatrix} A_n^0 \\ B_n^0 \\ C_n^0 \end{pmatrix} + \frac{1}{2} f_2(\mathcal{L}) \begin{pmatrix} A_n(t) \\ 2B_n(t) \\ 3C_n(t) \end{pmatrix}. \tag{4}$$

For each equation of the class (4) we are able to write down the M operator which provides us with the corresponding evolution of the eigenfunction of the L operator,

$$\frac{\mathrm{d}\psi_n}{\mathrm{d}t} = -M\psi_n. \tag{5}$$

In equation (4) we have

$$\begin{aligned} A_n^0 &= \gamma_0\left[B_n - B_{n+1} - A_n^2 + 2A_n\sum_{j=0}^{\infty}(-1)^j A_{n+j+1}\right] + \gamma_1[C_n - C_{n+2}] \\ &\quad + \beta_0(-1)^n\left[B_n - B_{n+1} + A_n^2 + 2A_n\sum_{j=0}^{\infty}A_{n+j+1}\right] - 2\beta_1(-1)^n A_n \\ B_n^0 &= \gamma_0[C_n - C_{n+1}] + \gamma_1[C_nA_{n-1} - C_{n+1}A_n] - \beta_0(-1)^n[C_n + C_{n+1}] \\ C_n^0 &= \gamma_0 C_n\left[A_{n-1} - 2\sum_{j=0}^{\infty}(-1)^j A_{n+j}\right] + \gamma_1 C_n[B_{n-1} - B_n] \\ &\quad + \beta_0(-1)^n C_n\left[A_n + 2\sum_{j=0}^{\infty}A_{n+j}\right] - 2\beta_1(-1)^n C_n \end{aligned} \tag{6}$$

where γ_0, γ_1, β_0 and β_1 are arbitrary constants, $f_1(z)$ and $f_2(z)$ are entire functions of their argument and the recursive operator $\mathcal{L}$ is given by a matrix of rank 3 whose components $\mathcal{L}_{i,j}$

with $(i, j = 1, 2, 3)$ are

$$\begin{aligned}
\mathcal{L}_{1,1} &= B_n\left[1+\sum_{j=1}^{\infty}E^{2j}\right]+A_n\sum_{j=1}^{\infty}\left[A_{n+2j-1}\sum_{k=0}^{\infty}E^{2k+2j}-A_{n+2j}\sum_{k=1}^{\infty}E^{2k+2j-1}\right]\\
&\qquad -B_{n+1}\sum_{j=1}^{\infty}E^{2j}-A_n\sum_{j=1}^{\infty}\left[A_{n+2j-2}\sum_{k=0}^{\infty}E^{2k+2j-1}-A_{n+2j-1}\sum_{k=0}^{\infty}E^{2k+2j-2}\right]\\
\mathcal{L}_{1,2} &= A_n\left[E+\sum_{j=1}^{\infty}E^{2j+1}-\sum_{j=1}^{\infty}E^{2j}\right]\\
\mathcal{L}_{1,3} &= E^2-C_{n+2}\sum_{j=1}^{\infty}(C_{n+j+1})^{-1}E^{j+1}+C_n\sum_{j=1}^{\infty}(C_{n+j-1})^{-1}E^{j-1}\\
\mathcal{L}_{2,1} &= C_n\left[E^{-1}+\sum_{j=1}^{\infty}E^{2j-1}\right]-C_{n+1}\sum_{j=1}^{\infty}E^{2j}\\
\mathcal{L}_{2,2} &= B_n\\
\mathcal{L}_{2,3} &= A_nE-A_nC_{n+1}\sum_{j=1}^{\infty}(C_{n+j})^{-1}E^{j}+A_{n-1}C_n\sum_{j=1}^{\infty}(C_{n+j-1})^{-1}E^{j-1},\\
\mathcal{L}_{3,1} &= C_n\sum_{j=1}^{\infty}\left[A_{n+2j-3}\sum_{k=0}^{\infty}E^{2k+2j-2}-A_{n+2j-2}\sum_{k=1}^{\infty}E^{2j+2k-3}-A_{n+2j-2}\sum_{k=0}^{\infty}E^{2j+2k-1}\right.\\
&\qquad \left.+A_{n+2j-1}\sum_{k=1}^{\infty}E^{2j+2k-2}\right]\\
\mathcal{L}_{3,2} &= C_n\left[E^{-1}+\sum_{j=1}^{\infty}E^{2j-1}-\sum_{j=1}^{\infty}E^{2j}\right]\\
\mathcal{L}_{3,3} &= B_n\left[1-C_n\sum_{j=1}^{\infty}(C_{n+j-1})^{-1}E^{j-1}\right]+B_{n-1}C_n\sum_{j=1}^{\infty}(C_{n+j-1})^{-1}E^{j-1}
\end{aligned} \tag{7}$$

If $f_2(\mathcal{L})$ is equal to zero than λ is constant and the corresponding equations (4) are isospectral deformations of the spectral problem (1), while if it is different from zero than $\lambda = \lambda(t)$ is given by

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = f_2(\lambda) \tag{8}$$

corresponding to non-isospectral deformations of (1).

The simplest equation of the hierarchy is obtained from (4) for $f_1(\mathcal{L}) = 1$, $f_2(\mathcal{L}) = 0$, $\gamma_0 = \beta_0 = \beta_1 = 0$ and $\gamma_1 = 1$, and it reads as:

$$\begin{aligned}
\frac{\mathrm{d}A_n(t)}{\mathrm{d}t} &= C_n(t) - C_{n+2}(t)\\
\frac{\mathrm{d}B_n(t)}{\mathrm{d}t} &= C_n(t)A_{n-1}(t) - C_{n+1}(t)A_n(t)\\
\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} &= C_n(t)[B_{n-1}(t) - B_n(t)]
\end{aligned} \tag{9}$$

In the literature this equation has been called the *Blaszak–Marciniak lattice* [11].

By applying the following transformation:

$$\begin{aligned}
C_n(t) &= \mathrm{e}^{(\alpha_{n+1}(t)-2\alpha_n(t)+\alpha_{n-1}(t))}\\
B_n(t) &= \frac{\mathrm{d}\alpha_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{n+1}(t)}{\mathrm{d}t}\\
A_n(t) &= \frac{\mathrm{d}^2\alpha_{n+1}}{\mathrm{d}t^2}\mathrm{e}^{(-\alpha_{n+2}(t)+2\alpha_{n+1}(t)-\alpha_n(t))}
\end{aligned} \tag{10}$$

we can rewrite equation (9) as an evolution equation of Toda type for the field $\alpha_n(t)$, which reads

$$\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3} = \frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}\left[\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t} - 2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}\right] + \mathrm{e}^{\alpha_{n-2}-\alpha_{n-1}-\alpha_n+\alpha_{n+1}} - \mathrm{e}^{\alpha_{n-1}-\alpha_n-\alpha_{n+1}+\alpha_{n+2}}. \tag{11}$$

We can extend equations (9) and (11) by considering non-isospectral deformations of the spectral problem (1). In this case, choosing $f_1(\mathcal{L}) = 1$, $f_2(\mathcal{L}) = 2a$, where a is a real constant, $\gamma_0 = \beta_0 = \beta_1 = 0$ and $\gamma_1 = 1$, equation (9) becomes

$$\begin{aligned}
\frac{\mathrm{d}A_n(t)}{\mathrm{d}t} &= C_n(t) - C_{n+2}(t) + aA_n(t)\\
\frac{\mathrm{d}B_n(t)}{\mathrm{d}t} &= C_n(t)A_{n-1}(t) - C_{n+1}(t)A_n(t) + 2aB_n(t)\\
\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} &= C_n(t)[B_{n-1}(t) - B_n(t) + 3a].
\end{aligned} \tag{12}$$

After applying the transformation

$$\begin{aligned}
C_n(t) &= \mathrm{e}^{(\alpha_{n+1}(t)-2\alpha_n(t)+\alpha_{n-1}(t))}\\
B_n(t) &= \frac{\mathrm{d}\alpha_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{n+1}(t)}{\mathrm{d}t} + 3an\\
A_n(t) &= \left[\frac{\mathrm{d}^2\alpha_{n+1}}{\mathrm{d}t^2} - 2a\frac{\mathrm{d}\alpha_{n+1}(t)}{\mathrm{d}t} + 3a^2n(n-1)\right]\mathrm{e}^{(-\alpha_{n+2}(t)+2\alpha_{n+1}(t)-\alpha_n(t))}
\end{aligned} \tag{13}$$

to equation (11) we obtain

$$\begin{aligned}
\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3} - 3a\frac{\mathrm{d}^2\alpha_n(t)}{\mathrm{d}t^2} + 2a^2\frac{\mathrm{d}\alpha_{n+1}(t)}{\mathrm{d}t} &= 3a^3(n-1)(n-2)\\
&\quad + \left[\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} - 2a\frac{\mathrm{d}\alpha_{n+1}(t)}{\mathrm{d}t} + 3a^2(n-1)(n-2)\right]\left[\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t} - 2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}\right]\\
&\quad + \mathrm{e}^{\alpha_{n-2}-\alpha_{n-1}-\alpha_n+\alpha_{n+1}} - \mathrm{e}^{\alpha_{n-1}-\alpha_n-\alpha_{n+1}+\alpha_{n+2}}.
\end{aligned} \tag{14}$$

Among the other possible choices of the coefficients and initial fields, one is particularly relevant and it corresponds to set $A_n = B_n = 0$ initially. This is the choice which in the case of the Toda hierarchy provides the Volterra equation [16]. In this case the admissible recursive operator is $\tilde{\mathcal{L}} = \mathcal{L}^3$ and the class of equations is given by

$$\begin{aligned}
\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} = \frac{3}{2}f_2(\tilde{\mathcal{L}})C_n(t) + f_1(\tilde{\mathcal{L}})\Bigg\{\gamma_0C_n(t)\sum_{j=0}^{\infty}[C_{n+2j-1}(t) - C_{n+2j}(t) - C_{n+2j+2}(t)\\
+ C_{n+2j+3}(t)] + \gamma_1C_n(t)[C_{n-1}(t)C_{n-2}(t) - C_{n+1}(t)C_{n+2}(t)]\Bigg\}
\end{aligned} \tag{15}$$

where f_1 and f_2 are entire functions of their argument and γ_0 and γ_1 are constant coefficients. As before, if $f_2 = 0$ the obtained equations correspond to isospectral deformation of the spectral problem (1), otherwise we have non-isospectral deformations.

The case when $\gamma_0 = 0$, $\gamma_1 = 1$ and $f_1(\tilde{\mathcal{L}}) = 1$, $f_2(\tilde{\mathcal{L}}) = 0$ provides the Volterra-like equation presented by Bogoyavlenskii [17]:

$$\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} = C_n(t)[C_{n-1}(t)C_{n-2}(t) - C_{n+1}(t)C_{n+2}(t)]. \tag{16}$$

Equation (16) can be extended by considering non-isospectral deformations of the corresponding spectral problem. In this case the simplest non-trivial equation is obtained by choosing $f_2 = \frac{2K}{3}$ and reads as

$$\frac{\mathrm{d}C_n(t)}{\mathrm{d}t} = C_n(t)[C_{n-1}(t)C_{n-2}(t) - C_{n+1}(t)C_{n+2}(t) + K]. \tag{17}$$

The generalized Lax formalism can be used to get the whole hierarchy of Bäcklund transformations, together with its recursion operator Λ. However, here we just present a Darboux transformation from which, taking into account the spectral problem, one can derive the Bäcklund transformation. By composition one can then construct all the higher-order Bäcklund transformations.

It is easy to prove that

$$\tilde{\alpha}_n(t) = \alpha_{n-1}(t) + \log(\psi_n(t;\lambda_1)) \tag{18}$$

$$\tilde{\psi}_n(t;\lambda) = \psi_{n-1}(t;\lambda)\frac{\psi_n(t;\lambda_1)}{\psi_{n-1}(t;\lambda_1)} - \psi_n(t;\lambda) \tag{19}$$

is a Darboux transformation for the Lax operator (1), once we use the transformation (10) to express $A_n(t)$, $B_n(t)$ and $C_n(t)$ in terms of $\alpha_n(t)$. Equations (18), (19) express the new field $\tilde{\alpha}_n(t)$ and the corresponding solution of the linear problem $\tilde{\psi}_n(t;\lambda)$ in terms of the old field $\alpha_n(t)$ and of the corresponding solution of the linear problem $\psi_n(t;\lambda)$ at the generic point λ of the spectrum and at a given point of the spectrum λ_1. Using equations (18), (19) we can construct a hierarchy of explicit solutions of equation (11). Starting from the trivial solution $\alpha_n(t) = 0$ from equations (10) we have that $B_n = 0$, $C_n = 1$ and $A_n = 0$. So, the 'naked' Lax pair reads

$$\begin{aligned} &\psi_{n+2}(t;\lambda) + \psi_{n-1}(t;\lambda) = \lambda\psi_n(t;\lambda) \\ &\frac{\mathrm{d}\psi_n(t;\lambda)}{\mathrm{d}t} = \psi_{n-1}(t;\lambda). \end{aligned} \tag{20}$$

Equations (20) can be solved explicitly. Defining $\lambda = z^2 + \frac{1}{z}$ we have

$$\psi_n(t;\lambda) = \sum_{j=1}^{3} \gamma_j z_j^n \mathrm{e}^{t/z_j} \tag{21}$$

where z_j is a solution of the third-order algebraic equation

$$z_j^3 - \lambda z_j + 1 = 0. \tag{22}$$

All the solutions of equation (22) can be expressed in terms of one arbitrary parameter (corresponding to the arbitrary value of λ), say z. So, we have

$$\begin{aligned} z_2 &= z \\ z_3 &= -\frac{1}{2}z\left[1 + \left(1 + \frac{4}{z^3}\right)^{1/2}\right] \\ z_1 &= -\frac{1}{2}z\left[1 - \left(1 + \frac{4}{z^3}\right)^{1/2}\right]. \end{aligned} \tag{23}$$

From equation (18) we get real solutions for $\tilde{\alpha}_n$ only if z is real and either $z > 0$ or $z < -2^{2/3}$. Morever, to get real solutions for $\tilde{\alpha}_n$, then $\tilde{\psi}_n(t;\lambda_1)$, given by equation (21), also has to be positive. This can be accomplished only if $\gamma_3 = 0$ when $z > 0$. In figure 1 we show the resulting solution.

Starting from the Darboux transformation (18), (19) we can construct a Bäcklund transformation by using the spectral problem (1). In terms of the variables α_n we have

$$\begin{aligned} &\mathrm{e}^{\tilde{\alpha}_{n+2}-\alpha_{n+1}} + \ddot{\alpha}_{n+1}\mathrm{e}^{2\alpha_{n+1}-2\alpha_n-\alpha_{n+2}+\tilde{\alpha}_{n+1}} + (\dot{\alpha}_n - \dot{\alpha}_{n+1})\mathrm{e}^{\tilde{\alpha}_n-\alpha_{n-1}} \\ &\qquad + \mathrm{e}^{\alpha_{n-1}-2\alpha_n+\alpha_{n+1}+\tilde{\alpha}_{n-1}-\alpha_{n-2}} = \lambda_1\mathrm{e}^{\tilde{\alpha}_n-\alpha_{n-1}}. \end{aligned} \tag{24}$$

To conclude, let us mention that equation (11) can be obtained not only for (2) but for any possible L of the third order. In fact, choosing in place of (2)

$$L = w_n(t)E^{-2} + v_n(t)E^{-1} + u_n(t) + E \tag{25}$$

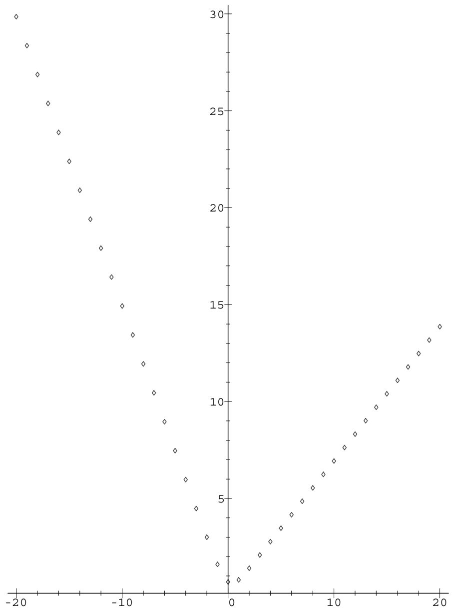

Figure 1. The first non-trivial exact solution of the differential–difference equation (11) $\alpha_n(t)$ for $z_2 = 2$, $\gamma_1 = 1$, $\gamma_2 = 1$, $\gamma_3 = 0$ and $t = 0$.

and defining

$$\begin{aligned} w_n(t) &= -\mathrm{e}^{(\alpha_{n+1}(t)-\alpha_n(t)-\alpha_{n-1}+\alpha_{n-2}(t))}, \\ v_n(t) &= \frac{\mathrm{d}^2\alpha_n(t)}{\mathrm{d}t^2} \\ u_n(t) &= \frac{\mathrm{d}\alpha_{n+1}(t)}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_n(t)}{\mathrm{d}t} \end{aligned} \qquad (26)$$

or choosing

$$L = w_n(t)E^{+2} + v_n(t) + u_n(t) + E^{-1} \qquad (27)$$

and defining

$$\begin{aligned} w_n(t) &= -\mathrm{e}^{(\alpha_{n+2}(t)-\alpha_{n+1}(t)-\alpha_n+\alpha_{n-1}(t))} \\ v_n(t) &= \frac{\mathrm{d}^2\alpha_n(t)}{\mathrm{d}t^2} \\ u_n(t) &= \frac{\mathrm{d}\alpha_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{n-1}(t)}{\mathrm{d}t} \end{aligned} \qquad (28)$$

or choosing

$$L = w_n(t)E^{-1} + v_n(t) + u_n(t)E + E^{-2} \qquad (29)$$

and defining

$$\begin{aligned} u_n(t) &= \mathrm{e}^{(\alpha_{n+1}(t)-2\alpha_n(t)+\alpha_{n-1}(t))} \\ v_n(t) &= \frac{\mathrm{d}\alpha_n(t)}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{n-1}(t)}{\mathrm{d}t} \\ w_n(t) &= \frac{\mathrm{d}^2\alpha_{n-1}}{\mathrm{d}t^2}\mathrm{e}^{(-\alpha_{n-2}(t)+2\alpha_{n-1}(t)-\alpha_n(t))} \end{aligned} \qquad (30)$$

we get always equation (11).

In conclusion, in this letter we have presented the recursion operator for a third-order discrete spectral problem. We have shown how the simplest system of equations of this hierarchy can be written down as a differential–difference equation of Toda type. By considering non-isospectral deformations of the third-order spectral problem we obtain a new class of differential–difference equations with n dependent coeffcients. Using a Darboux transformation we have written down an explicit solution for equation (11). We are currently looking for applications of equation (11) and to some extension of this work to higher-order spectral problems.

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